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Generalised geometry for M-theory

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Abstract: Generalised geometry studies structures on a d-dimensional manifold with a metric and 2-form gauge field on which there is a natural action of the group $\mathrm{SO}(d,d)$. This is generalised to d-dimensional manifolds with a metric and 3-form gauge field on which there is a natural action of the group E_d. This provides a framework for the discussion of M-theory solutions with flux. A different generalisation is to d-dimensional manifolds with a metric, 2-form gauge field and a set of p-forms for p either odd or even on which there is a natural action of the group E_{d+1}. This is useful for type IIA or IIB string solutions with flux. Further generalisations give extended tangent bundles and extended spin bundles relevant for non-geometric backgrounds. Special structures that arise for supersymmetric backgrounds are discussed.

Keywords: Flux compactifications, Superstring Vacua, M-Theory, String Duality.

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1. Introduction

Hitchin's generalised geometry [1]–[4] studies structures on a d-dimensional manifold M on which there is a natural action of the group $\mathrm{SO}(d, d)$, and in particular it gives an elegant description of geometries equipped with both a metric G and a 2-form B. Such geometries with a metric and 2-form and an action of $\mathrm{SO}(d, d)$ arise in string theory, so that this is a natural framework in which to formulate many problems in string theory and supergravity [5]–[20]. However, in type II string theory, the group $\mathrm{SO}(d,d)$ is part of a much larger 'U-duality' group [21, 22] E_{d+1} that acts on G and B together with a set of other fields on M (the Ramond-Ramond gauge fields) and this suggests that seeking a generalisation of generalised geometry with $\mathrm{SO}(d,d)$ replaced by E_{d+1} might provide a natural framework for the geometries with flux in type II string theory. Here E_n is the maximally non-compact real form of the group of rank-n with E-type Dynkin diagram, so that it is the exceptional group E_n for $n = 6, 7, 8$. The U-duality groups E_n and their maximal compact subgroups H_n are given in table 1 for $2 \leq n \leq 8$. These groups were found to be symmetries of supergravity theories in [21] and the global structure of these groups and their maximal subgroups was discussed in [23, 24].

M-theory has a similar structure in which there is a metric G and 3-form C on an n-dimensional manifold $\mathcal{M}$ with a natural action of E_n, and again one might expect a generalisation of generalised geometry with the 3-form C playing a central role. The relationship between M-theory and string theory suggests that if the manifold $\mathcal{M}$ is a circle bundle over a manifold M of dimension $d = n - 1$, then the M-geometry on $\mathcal{M}$ should reduce to a stringy generalised geometry on M.

The aim of this paper is to propose such generalisations, and to set up the framework needed to study general supersymmetric string or M-theory backgrounds, including non-geometric ones. This will lead to the introduction of new structures, and in particular to extended tangent bundles and extended spin bundles for type II geometries and M-geometries. It will be convenient to refer to the usual generalised geometry involving $\mathrm{SO}(d,d)$ as a type I geometry, to distinguish it from these other geometries, and it indeed plays a role in type I superstrings.

In generalised geometry, the tangent bundle T is replaced with $T \oplus T^*$, the sum of the tangent and cotangent bundles, which has a natural inner product of signature (d,d) preserved by the action of $\mathrm{SO}(d,d)$. This group includes the action of a 2-form on the geometry, which acts as a shift of B. A generalisation of the spin bundle is a bundle S over M with transition functions in $Spin(d,d)$. Given a choice of spin structure, there is a correspondence between S and the bundle $\Lambda^\bullet T^*$ of formal sums of differential forms on M, and S splits into the chiral and anti-chiral sub-bundles S^+ and S^- corresponding to even and odd forms respectively. The perturbative charges of string theory (momentum and string charge) fit into a vector of $\mathrm{SO}(d,d)$. In addition, there are Ramond-Ramond charges which are even forms for the type IIA string and odd forms for the IIB string, and the Ramond-Ramond charges transform according to the spinor representation of the $\mathrm{SO}(d,d)$ subgroup of the U-duality group [22]. This suggests that $T \oplus T^*$ be extended to $T \oplus T^* \oplus S^+$ for IIA or $T \oplus T^* \oplus S^-$ for IIB. This turns out to be sufficient for $d \leq 4$, but for $d > 4$ there are further charges consisting of a five-brane charge given by a 5-form in $\Lambda^5 T^*$ and a charge related to the Kaluza-Kein monopoles[1] [25] represented by a 5-vector in

[1]In $D = 10$ or $D = 11$, there is a 5-form charge in the superalgebra, $Z_{M_1 \dots M_5}$. Decomposing the indices $M = (0, i)$ where $i = 1, \dots, D-1$ is a spatial index and 0 is a time index gives two charges, a spatial 5-form charge $Z_{i_1 \dots i_5}$ which is the NS-NS or M-theory 5-brane charge, and a spatial 4-form charge $Z_{0 i_1 \dots i_4}$, which is the Kaluza-Kein monopole charge, Hodge-dual to a spatial $D-5$-vector $Z^{i_1 \cdots i_{D-5}}$ [25]. This gives

$\Lambda^5 T$, so that for type II strings the tangent bundle is generalised to the extended tangent bundle

$$T \oplus T^* \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus S^\pm$$

As will be seen in section 6, there is a natural action of E_{d+1} on this space for $d \leq 6$.

A bundle with structure group $O(d,d)$ is reducible to an $O(d) \times O(d)$ bundle. In generalised geometry, the metric G and 2-form B arise as the moduli for such reductions, and parameterise a coset space $O(d,d)/O(d) \times O(d)$. This is generalised to the coset E_{d+1}/H_{d+1} which can be parameterised by a metric G and 2-form B and scalar Φ, together with a set of odd forms $C_1, C_3, \ldots$ for IIA geometries or a set of even forms $C_0, C_2, C_4, \ldots$ for IIB geometries. These extra fields have a natural interpretation in type II string theory as the dilaton Φ and the Ramond-Ramond p-form gauge fields C_p. The formal sums $C^+ = C_0 + C_2 + C_4 + \cdots$ or $C^- = C_1 + C_3 + C_5 + \cdots$ transform as chiral spinors under $Spin(d,d)$, with the index $\pm$ indicating the chirality. The action of E_{d+1} on these fields includes shifts for each of the p-form gauge fields of the theory.

Comparison with M-theory suggests a different generalisation, replacing T^* (corresponding to a string charge) with $\Lambda^2 T^*$ (corresponding to a membrane charge), so that the extended tangent bundle includes $T \oplus \Lambda^2 T^*$. For manifolds of dimension $n > 4$, it is necessary to add $\Lambda^5 T^*$ (corresponding to a 5-brane charge), and for $n > 5$ an additional $\Lambda^6 T$ (the Kaluza-Kein monopole charge [25]) is needed. Then for $n \leq 7$, the extended tangent bundle is

$$T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$$

There is a natural action of E_n on this. The coset space E_n/H_n can be parameterised by a metric G, a 3-form C and (for $n \geq 6$) a 6-form $\widetilde{C}$ on the n-dimensional manifold. The 3-form C can be associated with the 3-form gauge field of 11-dimensional supergravity, and the 6-form $\widetilde{C}$ with the dual gauge field. (Recall that a free 3-form gauge field in 11-dimensions has a dual representation in terms of a 6-form gauge field, related by an electromagnetic duality, $d\widetilde{C}_6 \sim *dC_3$. The Chern-Simons interaction of 11-dimensional supergravity prevents the dualisation to a theory written in terms of a 6-form gauge field, but it can be written in terms of both a 3-form C and a 6-form $\widetilde{C}$, [26].) The action of E_n on these fields includes shifts of the 3-form field C and 6-form field $\widetilde{C}$.

For a d-dimensional manifold, the structure group of T, T^*, $T \oplus T^*$ (and their tensor products) is in $GL(d, \mathbb{R})$, which is a subgroup of $O(d,d)$. Twisting with a gerbe can enlarge the structure group to include the action of exact 2-forms [1, 2, 4], but this is still only a part of $O(d,d)$; this is the 'geometric subgroup' that preserves the Courant bracket. However, the covariance under the larger group $O(d,d)$ is very suggestive, and this suggests that bundles with this larger structure group might have an interesting role to play. String theory can in fact be formulated on a large class of spaces with so-called non-geometric structures, and including these allows a wider class of transition functions. For example, for string theory on a manifold M that is an m-torus bundle with fibres $\mathbf{T}^m$, there is a symmetry under the action of the T-duality group $O(m,m;\mathbb{Z})$, which in

charges in $\Lambda^5 T \oplus \Lambda^5 T^*$ for $D = 10$ or in $\Lambda^6 T \oplus \Lambda^5 T^*$ for $D = 11$.

n	E_n	H_n	$\dim(E_n)$	$\dim(E_n/H_n)$
2	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}$	$\mathrm{SO}(2)$	4	3
3	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	11	7
4	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	24	14
5	$Spin(5,5)$	$(\mathrm{Sp}(2)\times\mathrm{Sp}(2))/\mathbb{Z}_2$	45	25
6	$E_{6(6)}$	$\mathrm{Sp}(4)/\mathbb{Z}_2$	78	42
7	$E_{7(7)}$	$\mathrm{SU}(8)/\mathbb{Z}_2$	133	70
8	$E_{8(8)}$	$Spin(16)/\mathbb{Z}_2$	248	128

Table 1: The U-duality groups E_n, their maximal compact subgroups H_n, and the dimensions of E_n and the cosets E_n/H_n.

particular mixes the metric and B-field together. This symmetry allows the construction of T-folds. These are spaces built from patches which are each of the form $U_\alpha \times \mathbf{T}^m$ with U_α open sets in the base, and with transition functions that include $O(m,m;\mathbb{Z})$ T-duality transformations [27]. As the patching is through symmetries of the theory, it leads to consistent backgrounds of string theory. However, these are not manifolds equipped with tensor fields but are considerably more general. The generalised tangent bundle for such spaces has $O(d,d)$ transition functions not contained within the geometric subgroup. These have generalisations to U-folds with fibres $\mathbf{T}^m$ whose transistion functions include transformations in the U-duality group $E_{m+1}(\mathbb{Z})$ [27], and the extended geometries discussed here provide a natural framework to discuss these geometries. Examples of T-folds have been studied in [30]–[39].

2. Generalised geometry

2.1 The structure of generalised geometry

In Hitchin's generalised geometry, the tangent bundle T of a d-dimensional manifold M is replaced with $T \oplus T^*$, so that one considers the formal sum $V = v + \xi$ of a vector field v with components v^i $(i = 1, \ldots, d)$ and a one-form ξ with components ξ_i, which can be thought of as a vector with $2d$ components V^I

$$V^I = \begin{pmatrix} v^i \\ \xi_i \end{pmatrix}, \tag{2.1}$$

There is a natural inner product η of signature (d,d) defined by

$$\eta(v+\xi, v+\xi) = 2\iota_v \xi$$

where ι denotes the interior product, so that $\iota_v\xi = v^i\xi_i$. The metric has components η_{IJ} given by

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{2.2}$$

This is invariant under the orthogonal group $O(d,d)$, with V transforming in the vector representation $V \to gV$, where g is represented by a matrix $g^I{}_J$ satisfying

$$g^t \eta g = \eta \tag{2.3}$$

The Lie algebra of $O(d,d)$ consists of matrices with the block decomposition

$$\begin{pmatrix} A & \beta \\ \Theta & -A^t \end{pmatrix}, \tag{2.4}$$

Here $A^i{}_j$ is an arbitrary $d \times d$ matrix, and so is a generator of the $GL(d,\mathbb{R})$ subgroup of matrices g of the form

$$\begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix}, \tag{2.5}$$

for arbitrary invertible matrices $M^i{}_j$. The Θ_{ij} are components of a 2-form $\Theta \in \Lambda^2 T^*$ generating the group of matrices

$$\begin{pmatrix} \mathbb{1} & 0 \\ \Theta & \mathbb{1} \end{pmatrix}, \tag{2.6}$$

sending

$$v + \xi \mapsto v + \xi + \iota_v \Theta \tag{2.7}$$

while $\beta \in \Lambda^2 T$ is a generator of the group of matrices of the form

$$\begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}, \tag{2.8}$$

sending

$$v + \xi \mapsto v + \xi + \iota_\xi \beta \tag{2.9}$$

The 'geometric subgroup' $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$ generated by A, Θ of matrices of the form

$$\begin{pmatrix} M & 0 \\ \Theta & (M^t)^{-1} \end{pmatrix}, \tag{2.10}$$

will play a role in what follows.

There is a natural action of $Spin(d,d)$ on the bundle of formal sums of differential forms $\Lambda^\bullet T^*$ on M, so that interesting geometric structures can be formulated in terms of spinors. For each $V = v + \xi \in T \oplus T^*$, there is a map $\Gamma_V : \Lambda^\bullet T^* \to \Lambda^\bullet T^*$ such that

$$\Gamma_V : \phi \mapsto \iota_v \phi + \xi \wedge \phi$$

for any $\phi \in \Lambda^\bullet T^*$. These maps satisfy a Clifford algebra, with

$$\Gamma_V \Gamma_{V'} + \Gamma_{V'} \Gamma_V = -2\eta(V,V')\mathbb{1} \tag{2.11}$$

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The Clifford action on $\Lambda^\bullet T^*$ gives in particular a representation of $Spin(d,d)$ on $\Lambda^\bullet T^*$. The action of $GL(d,\mathbb{R}) \subset Spin(d,d)$ on $\Lambda^\bullet T^*$ is not quite the usual one. If the standard action of $M \in GL(d,\mathbb{R})$ on $\Lambda^\bullet T^*$ is denoted M^*, the action of $GL(d,\mathbb{R}) \subset Spin(d,d)$ is

$$\phi \mapsto |\det\, M|^{1/2} M^* \phi$$

so that the relation with the spin bundle S is

$$S = \Lambda^\bullet T^* \otimes (\Lambda^d T)^{1/2}$$

The bundle of forms splits into the bundle $\Lambda^+ T^*$ of even forms and the bundle $\Lambda^- T^*$ of odd forms, corresponding to the decomposition of S into bundles $S^\pm$ of positive or negative chirality spinors, with

$$S^\pm = \Lambda^\pm T^* \otimes (\Lambda^d T)^{1/2}$$

The bundle $(\Lambda^d T)^{1/2}$ is trivial and so there is always a non-canonical isomorphism $S^\pm \sim \Lambda^\pm T^*$; $S^\pm$ and $\Lambda^\pm T^*$ will be used interchangably for the remainder of the paper. (There is in addition another possible spin structure [4], but this will not be used here.)

The Courant bracket provides a generalisation of the Lie bracket to $T \oplus T^*$, and plays a central role in generalised geometry, and is preserved under (2.7) provided Θ is closed. According to Hitchin [2], generalised geometries are structures on $T \oplus T^*$ that are compatible with the SO(d,d) structure and which satisfy integrability conditions expressed in terms of the Courant bracket or the exterior derivative.

The transition functions for M are diffeomorphisms, so that the transition functions for $T \oplus T^*$ are in $GL(d,\mathbb{R})$, although it is sometimes useful to instead regard it as having structure group in SO(d,d) [4]. This can be generalised by twisting with a gerbe, as will be reviewed in the next subsection. For $d = 2m$, a generalised almost complex structure is an endomorphism $\mathcal{J}$ of $T \oplus T^*$ that satisfies $\mathcal{J}^2 = -\mathbb{1}$ and with respect to which the metric η is hermitian. It is a generalised complex structure if it is integrable, i.e. the $+i$-eigenbundle $E < (T \oplus T^*) \otimes \mathbb{C}$ is such that the space of sections of E is closed under the Courant bracket. Such a structure is preserved under the $U(m,m)$ subgroup of SO$(2m,2m)$.

Gualtieri introduced the concept of a generalised metric $\mathcal{H}$ on $T \oplus T^*$ [4]. This is a positive definite metric compatible with η, and defines a sub-bundle E_+ on which η is positive definite. The generalised metric can be represented by a matrix $\mathcal{H}_{IJ}$ satisfying the compatibility condition

$$\eta^{-1}\mathcal{H}\eta^{-1} = \mathcal{H}^{-1} \tag{2.12}$$

This implies that $S^I{}_J$ defined by

$$S = \eta^{-1}\mathcal{H} \tag{2.13}$$

satisfies

$$S^2 = \mathbb{1} \tag{2.14}$$

and so is an almost real structure or almost local product structure. (S is sometimes also referred to as the generalised metric [4].) It has d eigenvalues of $+1$ and d eigenvalues of -1, and E_+ is the $+1$ eigenbundle.

The constraint (2.12) implies that $\mathcal{H}$ has d^2 independent components and it can be parameterised in terms of a symmetric matrix G_{ij} and an anti-symmetric matrix B_{ij} as

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{2.15}$$

and $\mathcal{H}$ is positive definite if G is. The norm of the vector $V = v + \xi$ is then

$$\mathcal{H}(V,V) = G(v,v) + G^*(\xi + \iota_v B, \xi + \iota_v B) \tag{2.16}$$

where G^* is the metric on T^* given by the inverse of G and $(\iota_v B)_i = v^j B_{ji}$. Thus introducing a generalised metric is equivalent to introducing a positive definite metric G and a 2-form B on M. This can be generalised to a metric G of signature (p,q) on M, in which case the generalised metric given by (2.15) has signature $(2p, 2q)$.

Under an $\mathrm{SO}(d,d)$ transformation

$$\mathcal{H} \to g^t \mathcal{H} g \tag{2.17}$$

This corresponds to a fractional linear transformation of G, B. Defining the $d \times d$ matrix

$$E_{ij} = G_{ij} + B_{ij} \tag{2.18}$$

and decomposing g into $d \times d$ matrices a, b, c, d

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.19}$$

so that

$$g^t \eta g = \eta \quad \Rightarrow \quad a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = \mathbb{1}, \tag{2.20}$$

then the transformation of G, B under the action of the $\mathrm{SO}(d,d)$ transformation g is

$$E' = (aE + b)(cE + d)^{-1}. \tag{2.21}$$

In particular, the action of the $GL(d, \mathbb{R})$ subgroup (2.5) is the linear transformation

$$G \to M^t G M, \qquad B \to M^t B M, \tag{2.22}$$

while the Θ transformation (2.6) leaves G invariant and acts as a shift of B:

$$B \to B + \Theta \tag{2.23}$$

However, $\mathrm{SO}(d,d)$ transformations not in the geometric subgroup will mix G and B.

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2.2 Gerbes and the generalised tangent bundle

For $T \oplus T^*$, the structure group is $GL(d, \mathbb{R})$ and introducing a generalised metric corresponds to introducing a symmetric tensor field G and an anti-symmetric tensor field B on M. However, this can be generalised to allow B to be a gerbe conection, i.e. a 2-form gauge field with field strength $H = dB$, allowing a twisting of this construction to allow transition functions including the B-shift.

Given an open cover $\{U_\alpha\}$ of M, there is a 2-form B_α in each $\{U_\alpha\}$ with $B_\beta - B_\alpha$ a closed 2-form on the overlap $U_\alpha \cap U_\beta$, so that $dB_\beta = dB_\alpha = H$ is a globally defined closed three-form H. For a suitable open cover, the overlaps have trivial cohomology and

$$B_\beta - B_\alpha = d\lambda_{\alpha\beta}$$

for some 1-form $\lambda_{\alpha\beta}$ on the overlap $U_\alpha \cap U_\beta$. Consistency on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ requires that $\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha}$ is closed and so exact. If it is of the form

$$\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$$

for some $\mathrm{U}(1)$-valued functions

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$$

satisfying $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma}$ and $g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$, then B_α defines a connection on a gerbe and H represents an integral cohomology class. (If H is not in an integral cohomology class, then

$$\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = d\rho_{\alpha\beta\gamma}$$

for some 0-form $\rho_{\alpha\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$ satisfying a further consistency condition in quadruple overlaps.)

The $\lambda_{\alpha\beta}$ can be used to define a bundle E over M by identifying $T \oplus T^*$ on U_α with $T \oplus T^*$ on U_β by the B-field action

$$v + \xi \mapsto v + \xi + \iota_v d\lambda_{\alpha\beta}$$

The fibre over a point x in M is again $T_x \oplus T^*_x$, but the transition functions are no longer in $GL(d, \mathbb{R})$. The bundle E has been called a generalised tangent bundle [2] and has a structure group in the geometric subgroup of $\mathrm{SO}(d, d)$, i.e. the subgroup $GL(d, \mathbb{R}) \ltimes \Omega^{2,cl}$, where $\Omega^{2,cl}$ is the space of closed 2-forms.

3. The structure of extended geometries

3.1 Type I extended geometries: generalising the generalised tangent bundle and spin bundle

To incorporate structures such as T-folds and other non-geometric backgrounds, it is useful to generalise the structure further and consider general bundles E over a d-dimensional space M with structure group $O(d,d)$ or $\mathrm{SO}(d,d)$ and split-signature fibre metric η; these will be generalised geometries in the sense of Hitchin only in the special case in which the structure group is in the geometric subgroup preserving the Courant bracket, and will only correspond to $T \oplus T^*$ if the structure group is in $GL(d,\mathbb{R})$. Locally, one can find a metric G and 2-form B as before, but general $O(d,d)$ transition functions mix G and B, so that these will not be tensor fields on M in general, and the background will be 'non-geometric'. Nonetheless, such backgrounds with m-torus fibrations and transition functions including $O(m,m;\mathbb{Z})$ transformations arise in string theory as T-folds, so that this is a useful generalisation. Such extended geometries with $O(d,d)$ structure will be referred to as Type I extended geometries, to distinguish them from the type II and type M geometries with E-series structure groups to be introduced later. It will also be natural to introduce an extended spin bundle S with structure group $Pin(d,d)$ or $Spin(d,d)$, when there is no obstruction to such a double cover of E.

The bundle E can be reduced to one that has structure group in the maximal compact subgroup $O(d)\times O(d)$ or $S(O(d)\times O(d))$. This is equivalent to choosing a sub-bundle E^+ on which η is positive definite, so that $E = E^+ \oplus E^-$ where E^- is the orthogonal complement of E^+, so that η is negative definite on E^-. An $\mathrm{SO}(d,d)$ bundle E admits a $Spin(d,d)$ structure only if the second Stiefel-Whitney classes of $E^\pm$ agree, $w_2(E^+) = w_2(E^-)$ [4, 48]; this is automatically satisfied for $T \oplus T^*$, even in the case in which M is not spin, i.e. even if $w_2(T) \neq 0$.

The reduction of E to $E^\pm$ defines a positive definite generalised metric

$$\mathcal{H} = \eta|_{E^+} - \eta|_{E^-} \tag{3.1}$$

Choosing a generalised metric is equivalent to choosing a reduction of the bundle, and the space of such reductions at a point $x \in M$ is

$$\frac{O(d,d)}{O(d)\times O(d)} \quad \text{or} \quad \frac{\mathrm{SO}(d,d)}{S(O(d)\times \mathrm{SO}(d))} \tag{3.2}$$

Let $\mathcal{V}^\pm$ be the projections $\mathcal{V}^\pm : E \to E^\pm$. Then

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}^+ \\ \mathcal{V}- \end{pmatrix} \tag{3.3}$$

maps $E \to E^+ \oplus E^-$ and is a representative of the coset $O(d,d)/O(d)\times O(d)$. Introducing indices $a = 1,\ldots,d$ labelling a basis for E^+ transforming under one $O(d)$ factor and indices $a' = 1,\ldots,d$ labelling a basis for E^- transforming under the other $O(d)$ factor, $\mathcal{V}^+$ is represented by a $d\times 2d$ matrix $\mathcal{V}^a{}_I$ and $\mathcal{V}^-$ is represented by a $d \times 2d$ matrix $\mathcal{V}^{a'}{}_I$, so that

$$\mathcal{V}^A{}_I = \begin{pmatrix} \mathcal{V}^a{}_I \\ \mathcal{V}^{a'}{}_I \end{pmatrix}, \tag{3.4}$$

is a vielbein transforming from a general basis labelled by I to a basis for $E^+ \oplus E^-$ labelled by $A = (a, a')$. The generalised metric is then

$$\mathcal{H} = \mathcal{V}^t\mathcal{V} \tag{3.5}$$

with components

$$\mathcal{H}_{IJ} = \delta_{AB}\mathcal{V}^A{}_I\mathcal{V}^B{}_J \tag{3.6}$$

The generalised metric is not constant over M in general, so $\mathcal{H}(x)$ (where $x \in M$) defines a map $\mathcal{H} : M \to O(d,d)/O(d) \times O(d)$. As well as the manifest covariance under $O(d,d)$, there is a symmetry under local $O(d) \times O(d)$ transformations, given by functions $k(x)$, with $k : M \to O(d) \times O(d)$. In particular, the vielbein $\mathcal{V}(x)$ transforms as

$$\mathcal{V}(x) \to k(x)\mathcal{V}(x)g \tag{3.7}$$

under a local $O(d) \times O(d)$ transformation $k(x)$ and rigid transformation $g \in O(d,d)$. The local $O(d) \times O(d)$ symmetry can be used to choose a triangular gauge for $\mathcal{V}$ over some neighbourghood of M, so that

$$\mathcal{V} = \begin{pmatrix} e^t & 0 \\ -e^{-1}B & e^{-1} \end{pmatrix}, \tag{3.8}$$

for some d-bein $e_i{}^a$ and anti-symmetric $d \times d$ matrix B_{ij}. Then

$$\mathcal{H} = \mathcal{V}^t\mathcal{V} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{3.9}$$

where the metric $G = e^t e$, i.e.

$$G_{ij} = e_i{}^a e_j{}^b \delta_{ab} \tag{3.10}$$

As a result, the fibre metric $\mathcal{H}(x)$ is parameterized by a $d \times d$ matrix $E(x)$ given by

$$E_{ij} = G_{ij} + B_{ij} \tag{3.11}$$

3.2 General extended geometries

The above structure generalises to arbitrary vector bundles with non-compact structure group G. Consider a vector bundle $\mathcal{E}$ over a manifold M with projection $\pi : \mathcal{E} \to M$, fibre F and structure group G. For an open cover $\{U_\alpha\}$ of M, $\pi^{-1}(U_\alpha) \sim U_\alpha \times F$ and a point in $\pi^{-1}(U_\alpha)$ can be represented by (x_α, V_α) where $x_\alpha \in U_\alpha$, $V_\alpha \in F$. The group G acts as $(x,V) \to (x, gV)$, where $gV \equiv R(g)V$ and $R(g)$ is the action of $g \in G$ on F in some representation R. Over the overlap $U_\alpha \cap U_\beta$, the coordinates in $\pi^{-1}(U_\alpha \cap U_\beta)$ are related by

$$V_\alpha = g_{\alpha\beta}(x)V_\beta \tag{3.12}$$

where the transition function $g_{\alpha\beta}(x)$ is a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ acting on F (and satisfying the usual consistency conditions).

For any maps $\mathcal{V}_\alpha : U_\alpha \to G$, the transition functions

$$h_{\alpha\beta} = \mathcal{V}_\alpha g_{\alpha\beta}\mathcal{V}_\beta^{-1} \tag{3.13}$$

define a bundle equivalent to $\mathcal{E}$. If G is non-compact with maximal compact subgroup H, then $\mathcal{E}$ can be reduced to a bundle $\bar{\mathcal{E}}$ with structure group H. This means that the maps

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$\mathcal{V}_\alpha : U_\alpha \to G$ can be chosen so that the transition functions (3.13) are in H, $h_{\alpha\beta} \in H$. For any such maps $\mathcal{V}_\alpha$, the maps $\mathcal{V}'_\alpha = h_\alpha \mathcal{V}_\alpha$ will also give transition functions in H, provided that h_α are maps $h_\alpha : U_\alpha \to H$. Then a reduction corresponds to an equivalence class of maps $\mathcal{V}_\alpha$ identified under the left action of maps $h_\alpha : U_\alpha \to H$, $\mathcal{V}_\alpha \sim h_\alpha \mathcal{V}_\alpha$. The equivalence classes then correspond to maps from U_α to the left coset G/H. From (3.13), the maps $\mathcal{V}_\alpha$ have the patching conditions

$$\mathcal{V}_\alpha = h_{\alpha\beta} \mathcal{V}_\beta g_{\alpha\beta}^{-1} \tag{3.14}$$

There is then a map

$$\mathcal{V} : \mathcal{E} \to \bar{\mathcal{E}}, \qquad \mathcal{V} : (x_\alpha, V_\alpha) \to (x_\alpha, \bar{V}_\alpha) \equiv (x_\alpha, \mathcal{V}_\alpha(x_\alpha))$$

where the $\bar{V}_\alpha = \mathcal{V}_\alpha(x_\alpha) V_\alpha$ have patching conditions at x

$$\bar{V}_\alpha = h_{\alpha\beta}(x) \bar{V}_\beta \tag{3.15}$$

with transition functions $h_{\alpha\beta} \in H$, so that $\bar{\mathcal{E}}$ is indeed a vector bundle with structure group H.

Suppose that the representation R has an H-invariant positive definite metric, giving a positive definite fibre metric $\bar{\mathcal{H}}(\bar{s}, \bar{s})$ for sections $\bar{s}(x)$ of $\bar{\mathcal{E}}$, and this in turn defines a positive definite fibre metric for sections $s(x)$ of $\mathcal{E}$, via

$$\mathcal{H}(s, s) = \bar{\mathcal{H}}(\mathcal{V}s, \mathcal{V}s) \tag{3.16}$$

For example, if H is an orthogonal group with $h^t h = \mathbb{1}$ where h^t is the transpose, then $\bar{\mathcal{H}}(\mathcal{V}s, \mathcal{V}s) = \bar{s}^t \bar{s}$ and $\mathcal{H}(s, s) = s^t \mathcal{H} s$ where the matrix $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{V}^t \mathcal{V} \tag{3.17}$$

For $G = O(d, d)$, this gives the $O(d) \times O(d)$ invariant metric (2.15). Similarly, for unitary groups with $h^\dagger h = \mathbb{1}$,

$$\mathcal{H} = \mathcal{V}^\dagger \mathcal{V} \tag{3.18}$$

There is a natural action of H gauge transformations, i.e. of maps $h_\alpha : U_\alpha \to H$ under which

$$\mathcal{V}_\alpha(x) \to h_\alpha(x) \mathcal{V}_\alpha, \qquad (x, \bar{V}_\alpha) \to (x, h_\alpha(x) \bar{V}_\alpha), \qquad h_{\alpha\beta} \to h_\alpha h_{\alpha\beta} h_\beta^{-1} \tag{3.19}$$

We will be interested in gauge equivalence classes identified under this action. In particular, the metric $\mathcal{H}$ depends only on the equivalence class, and so is specified by a map $M \to G/H$, or more generally a section of a bundle with fibre G/H.

Finally, for many cases of interest, H has a natural double cover $\widetilde{H}$, and so given the extended tangent bundle $\bar{\mathcal{E}}$ with H-structure, it is natural to seek an extended spin-bundle $\widetilde{\mathcal{E}}$ with structure group $\widetilde{H}$ that projects onto $\bar{\mathcal{E}}$ under the double cover map $p : \widetilde{H} \to H$. There is in general a topological obstruction for such a double cover, given by the 2nd Stiefel-Whitney class $w_2(\bar{\mathcal{E}}) = H^2(\bar{\mathcal{E}}, \mathbb{Z}_2)$. Given a lift of the transition functions $h_{\alpha\beta}$ to

$\widetilde{h}_{\alpha\beta} \in \widetilde{H}$, the $\mathbb{Z}_2$ Cech cohomology class is represented by the $\widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\gamma\alpha} = \pm\mathbb{1}$ in triple overlaps, and it is necessary to be able to choose the $\widetilde{h}_{\alpha\beta}$ so that this is $+\mathbb{1}$ in all triple overlaps. A necessary and sufficient condition for this is that $w_2(\bar{\mathcal{E}}) = 0$.

In the following sections, examples of this construction with $G = E_n$ and $H = H_n$ will be explored.

4. M-geometries

In this section, the generalisation of generalised geometry suggested by M-theory on an orientible n-dimensional manifold $\mathcal{M}$ are investigated, in which $T \oplus T^*$ with a natural action of $\mathrm{SO}(n,n)$ is replaced by $\mathcal{E} \sim T \oplus \Lambda^2 T^* \oplus \dots$ with a natural action of E_n, and the 2-form symmetry of B-shifts is generalised to one of 3-form shifts. The structure changes from dimension to dimension, so each will be considered in turn. The full explicit transformations will be given only for $n = 4, 7$; those for $n = 5, 6$ follow by truncation of the $n = 7$ case.

4.1 $n = 4$, $E_4 = \mathrm{SL}(5, \mathbb{R})$

Consider first the case of a four manifold, with $E_4 = \mathrm{SL}(5, \mathbb{R})$. The bundle $T \oplus T^*$ is replaced with $T \oplus \Lambda^2 T^*$ with 10-dimensional fibres transforming in the $4 + 6$ representation of $\mathrm{SL}(4, \mathbb{R})$. A section is then a formal sum

$$U = v + \rho$$

of a vector v and a 2-form ρ which can be thought of as an extended vector with 10 components U^I ($I = 1, \dots, 10$)

$$U^I = \begin{pmatrix} v^i \\ \rho_{ij} \end{pmatrix}, \tag{4.1}$$

where $i, j = 1, \dots, 4$ and $\rho_{ij} = -\rho_{ji}$.

There is an action of $\mathrm{SL}(5, \mathbb{R})$ on $T \oplus \Lambda^2 T^*$, as follows. First, there is the natural action of $\mathrm{SL}(4, \mathbb{R})$ acting separately on the vector v and 2-form ρ. There is an action of a 3-form $\Theta \in \Lambda^3 T^*$ sending

$$v + \rho \mapsto v + \rho + \iota_v \Theta \tag{4.2}$$

and the action of a tri-vector $\beta \in \Lambda^3 T$ with components β^{ijk} sending

$$v + \rho \mapsto v + \rho + \iota_\rho \beta$$

(with $(\iota_\rho \beta)^i = \frac{1}{2}\rho_{jk}\beta^{jki}$). These are natural generalisations of (2.7), (2.9). Finally, the group closes on a scaling under which

$$v + \rho \mapsto \alpha^3 v + \alpha^2 \rho \tag{4.3}$$

with $\alpha \in \mathbb{R}, \alpha \neq 0$. The adjoint of $\mathrm{SL}(5, \mathbb{R})$ decomposes as

$$24 = 15 + 1 + 4 + 4'$$

under $\mathrm{SL}(4,\mathbb{R})$, corresponding to these four classes of transformation. The fibres then transform in the 10-dimensional representation of $\mathrm{SL}(5,\mathbb{R})$ labelled by the index $I = 1, \ldots, 10$.

An $\mathrm{SL}(5,\mathbb{R})$ bundle $\mathcal{E}$ can be reduced to an $\mathrm{SO}(5)$ bundle $\bar{\mathcal{E}}$, and the reduction is equivalent to choosing an element $\mathcal{V}$ of the coset $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)$, or equivalently a positive definite fibre metric $\mathcal{H}$, for each point $x \in \mathcal{M}$. This can be represented by a matrix function $\mathcal{V}^A{}_I(x)$ on some patch $U \subset \mathcal{M}$ where $A = 1, \ldots, 10$ labels the 10-dimensional representation of $\mathrm{SO}(5)$. Given a metric G_{ij} and orientation on $\mathcal{M}$, the tangent bundle becomes an $\mathrm{SO}(4)$ bundle whose structure group is a subgroup of the $\mathrm{SO}(5)$, and the 10-dimensional representation decomposes as $10 = 4 + 6$ under $\mathrm{SO}(4) \subset \mathrm{SO}(5)$.

The coset $\mathrm{SL}(5,\mathbb{R})/\mathrm{SO}(5)$ is 14-dimensional and can be parameterised by a symmetric matrix G_{ij} transforming in the $\mathbf{10}$ of $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ and a 3-form C_{ijk} transforming as a $\mathbf{4}$ of $\mathrm{SO}(4)$. At each point $x \in \mathcal{M}$, the vielbein $\mathcal{V}(x)$ transforms as

$$\mathcal{V}(x) \to k(x)\mathcal{V}(x)g \tag{4.4}$$

under a local $\mathrm{SO}(5)$ transformation $k(x)$ and rigid transformation $g \in \mathrm{SL}(5,\mathbb{R})$. It is useful to introduce a frame field $e^a{}_i$ for $T\mathcal{M}$, so that $G_{ij} = \delta_{ab}e^a{}_i e^b{}_j$ with tangent space indices $a, b\ldots$ transforming under $\mathrm{SO}(4)$, and the vielbein $e^a{}_i$ is used to convert indices $i, j\ldots$ to $a, b\ldots$, so that e.g. $v^a = e^a{}_i v^i$. The local $\mathrm{SO}(5)$ symmetry can be used to choose a triangular gauge for $\mathcal{V}$ over some neighbourhood of $\mathcal{M}$, so that

$$\mathcal{V} = \begin{pmatrix} e^a{}_i & 0 \\ -e^j_a e_b{}^k C_{ijk} & e^{[i}_a e_b{}^{j]} \end{pmatrix} \tag{4.5}$$

It maps U given by (4.1) to

$$\bar{U}^A = \begin{pmatrix} u^a \\ u_{ab} \end{pmatrix} = \mathcal{V}^A{}_I U^I = \begin{pmatrix} v^a \\ \rho_{ab} - C_{abc}v^c \end{pmatrix} \tag{4.6}$$

An $\mathrm{SO}(5)$-invariant metric on sections of $\bar{\mathcal{E}}$ is given by

$$\bar{\mathcal{H}}(\bar{U}, \bar{U}) = \bar{\mathcal{H}}_{AB}\bar{U}^A\bar{U}^B = \delta_{ab}u^a u^b + \frac{1}{2}\delta^{ab}\delta^{cd}u_{ac}u_{bd} \tag{4.7}$$

Then a positive definite generalised metric $\mathcal{H}$ on $\mathcal{E}$ can be defined by (3.16) giving the norm of (4.1) as

$$\mathcal{H}(U, U) = G(v, v) + G^*(\rho - \iota_v C, \rho - \iota_v C) \tag{4.8}$$

where G^* is the norm on 2-forms constructed from $G = e^t e$. In terms of components, this is

$$\mathcal{H}(U, U) = G_{ij}v^i v^j + \frac{1}{2}G^{ik}G^{jl}(\rho_{ij} - C_{ijm}v^m)(\rho_{kl} - C_{kln}v^n) \tag{4.9}$$

so that the metric is represented by the matrix $\mathcal{H} = \mathcal{V}^t\bar{\mathcal{H}}\mathcal{V}$ which has the form

$$\mathcal{H} = \begin{pmatrix} G + \frac{1}{2}CG^{-1}G^{-1}C & -\frac{1}{2}CG^{-1}G^{-1} \\ -\frac{1}{2}G^{-1}G^{-1}C & \frac{1}{2}G^{-1}G^{-1} \end{pmatrix} \tag{4.10}$$

The action of the 3-form transformation on $\mathcal{V}$ and $\mathcal{H}$ gives

$$C \mapsto C + \Theta \tag{4.11}$$

so that the three-form transformation shifts the three-form field C.

4.2 $n = 5$, $E_5 = Spin(5,5)$

Consider next the case of a five-manifold, with $E_5 = Spin(5,5)$. In this case, in addition to the 2-form, a 5-form is added to the fibres. The bundle $T \oplus T^*$ is then replaced with $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*$ with 16-dimensional fibres transforming in the $5 + 10' + 1$ representation of $\mathrm{SL}(5,\mathbb{R})$. A section is then a formal sum

$$U = v + \rho + \sigma$$

of a vector v, a 2-form ρ and a 5-form σ. Given a volume form $\epsilon \in \Lambda^5 T^*$ and its dual $\widetilde{\epsilon} \in \Lambda^5 T$ with $\iota_{\widetilde{\epsilon}}\epsilon = 1$, this is equivalent to to the sum of a 0-form $*\sigma = \iota_{\widetilde{\epsilon}}\sigma$, a 2-form ρ and a 4-form $*v = \iota_v \epsilon$, and so there is a natural action of $Spin(5,5)$ on this under which the fibres transform as $\mathbf{16}^+$, the positive chirality spinor representation. The adjoint of $Spin(5,5)$ decomposes under $\mathrm{SL}(5,\mathbb{R})$ as

$$\mathbf{45} = \mathbf{24} + \mathbf{1} + \mathbf{10} + \mathbf{10}'$$

consisting of the natural action of $\mathrm{SL}(5,\mathbb{R})$ on tangent vectors and forms on a 5-fold, a scaling transformation and the action of a 3-form Θ_{ijk} and a 3-vector β^{ijk}, so that this is very similar to the $n = 4$ case. The coset space $Spin(5,5)/H_5$ where $H_5 = (Spin(5) \times Spin(5))/\mathbb{Z}_2$ has dimension 25 and can be parameterised by a symmetric matrix G_{ij} and 3-form C_{ijk}. Then as for $n = 4$, there is a generalised metric $\mathcal{H}(x)$ and vielbein $\mathcal{V}$ parameterised by a metric $G_{ij}(x)$ and 3-form $C_{ijk}(x)$ on $\mathcal{M}$, with the 3-form transforming as $C \mapsto C + \Theta$.

4.3 $n = 6$, E_6

As for $n = 5$, the bundle $T \oplus T^*$ is replaced with $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^*$ with 27-dimensional fibres transforming in the $6 + 15 + 6$ representation of $\mathrm{SL}(6,\mathbb{R})$, with a natural action of E_6 acting in the $\mathbf{27}$ representation. A section is a 27-dimensional vector decomposing as a formal sum

$$U = v + \rho + \sigma$$

of a vector v, a 2-form ρ and a 5-form σ. The adjoint of E_6 decomposes under $\mathrm{SL}(6,\mathbb{R})$ as

$$\mathbf{78} = \mathbf{35} + \mathbf{1} + \mathbf{20} + \mathbf{20} + \mathbf{1} + \mathbf{1}$$

consisting of the natural action of $\mathrm{SL}(5,\mathbb{R})$ on tangent vectors and forms on a 5-fold, a scaling transformation, the action of a 3-form Θ_{ijk} and a 3-vector β^{ijk}, as before, but now in addition there is the action of a 6-form $\widetilde{\Theta} \in \Lambda^6 T^*$ and a 6-vector $\widetilde{\beta} \in \Lambda^6 T$; these are singlets, but regarding them as 6-forms and 6-vectors is suggested by the fact that 6-forms and 6-vectors arise for $n = 7$. The coset E_6/H_6 where $H_6 = \mathrm{Sp}(4)/\mathbb{Z}_2$ is 42-dimensional and can be parameterised by a symmetric matrix G_{ij}, a 3-form C_{ijk} and a 6-form $\widetilde{C}_{i_1 \dots i_6}$ (dual to a scalar in 6 dimensions). Then the generalised metric $\mathcal{H}(x)$ and vielbein $\mathcal{V}$ are parameterised by a metric $G_{ij}(x)$, 3-form $C_{ijk}(x)$ and a 6-form $\widetilde{C}_{i_1 \dots i_6}(x)$ on $\mathcal{M}$. The group E_6 has a maximal subgroup $\mathrm{SL}(6,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ under which

$$\mathbf{27} \rightarrow (\mathbf{6},\mathbf{2}) + (\mathbf{15},\mathbf{1}), \qquad \mathbf{78} \rightarrow (\mathbf{35},\mathbf{1}) + (\mathbf{20},\mathbf{2}) + (\mathbf{1},\mathbf{3})$$

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For $n = 7$, as will be seen below, the bundle $T \oplus T^*$ is replaced with $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$, suggesting that for $n = 6$ one also consider a generalisation in which $\Lambda^6 T$ is added to the generalised tangent bundle. Then $\Lambda^6 T$ is invariant under E_6 and $\mathrm{SL}(6, \mathbb{R})$, so that E_6 acts on $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$ in the $\mathbf{27} + \mathbf{1}$ representation. This extra singlet corresponds to an extra charge that is allowed by the supersymmetry algebra [49]. It is not known whether states carrying this charge arise in M-theory, but if they do, their presence would have dramatic implications [50].

4.4 $n = 7$, E_7

For $n = 7$, the bundle $T \oplus T^*$ is replaced with

$$T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$$

with 56-dimensional fibres transforming in the $7 + 21' + 21 + 7'$ representation of $\mathrm{SL}(7, \mathbb{R})$, with a natural action of E_7 acting in the $\mathbf{56}$ representation. E_7 has a maximal $\mathrm{SL}(8, \mathbb{R})$ subgroup, and these $\mathrm{SL}(7, \mathbb{R})$ representations combine into the $28 + 28'$ of $\mathrm{SL}(8, \mathbb{R})$. A section is a 56-dimensional vector decomposing as a formal sum

$$U = v + \rho + \sigma + \tau$$

of a vector v, a 2-form ρ, a 5-form σ and a 6-vector τ.

The adjoint of E_7 decomposes under $\mathrm{SL}(7, \mathbb{R})$ as

$$\mathbf{133} = \mathbf{48} + \mathbf{1} + \mathbf{35} + \mathbf{35}' + \mathbf{7} + \mathbf{7}'$$

and so in addition to the standard action of $\mathrm{SL}(7, \mathbb{R})$ and a scaling, there is the action of a 3-form $\Theta \in \Lambda^3 T^*$, a 3-vector $\beta \in \Lambda^3 T$, a 6-form $\widetilde{\Theta} \in \Lambda^6 T^*$ and a 6-vector $\widetilde{\beta} \in \Lambda^6 T$. The action of the 6-form and 6-vector combine with the action of $\mathrm{SL}(7, \mathbb{R})$ and the scaling to generate an $\mathrm{SL}(8, \mathbb{R})$ subgroup. The coset E_7/H_7 where $H_7 = \mathrm{SU}(8)/\mathbb{Z}_2$ is 70-dimensional and can be parameterised by a symmetric matrix G_{ij}, a 3-form C_{ijk} and a 6-form $\widetilde{C}_{i_1 \dots i_6}$ ($70 = 28 + 35 + 7$). Then the generalised metric $\mathcal{H}(x)$ and vielbein $\mathcal{V}$ is specified in terms of a metric $G_{ij}(x)$, 3-form $C_{ijk}(x)$ and a 6-form $\widetilde{C}_{i_1 \dots i_6}(x)$ on $\mathcal{M}$.

The action of E_7 can be understood as follows. Consider the 8-manifold $N = \mathcal{M} \times S^1$ with the natural $\mathrm{U}(1)$ action generated by a vector k tangent to S^1; let $\widetilde{k}$ be the dual one-form on S^1, with $\widetilde{k}(k) = 1$. If $\theta \sim \theta + 2\pi$ is the S^1 coordinate, then $k = \partial/\partial\theta$ and $\widetilde{k} = d\theta$. A 2-form ϕ on N is specified by a 1-form $\phi_1 = \iota_k \phi$ and a 2-form $\phi_2 = \phi - \widetilde{k} \wedge \iota_k \phi$ with $\iota_k \phi_2 = 0$, and if ϕ is $\mathrm{U}(1)$ invariant (i.e. the Lie derivative $\mathcal{L}_k \phi = 0$) these pull-back to a 1-form ϕ_1' and 2-form ϕ_2' on $\mathcal{M}$. We can then define a 2-form and 6-vector on $\mathcal{M}$ by $\rho = \phi_2'$ and $\tau = *\phi_1' = \iota_{\phi_1'} \widetilde{\epsilon}$ where $\widetilde{\epsilon}$ is the 7-vector on $\mathcal{M}$ dual to the volume form ϵ. The natural action of $\mathrm{SL}(8, \mathbb{R})$ on $\Lambda^2 T^* N$ gives the action of $\mathrm{SL}(8, \mathbb{R})$ on ϕ and hence on ρ, τ which transform according to the $\mathbf{28}'$ representation. Similarly, an invariant bi-vector $\chi \in \Lambda^2 TN$ gives a vector $\chi_1' \in T\mathcal{M}$ and a bi-vector $\chi_2' \in \Lambda^2 T\mathcal{M}$, and these define a vector $v = \chi_1'$ and a 5-form $\sigma = *\chi_2' = \iota_{\chi_2'} \epsilon$. The action of $\mathrm{SL}(8, \mathbb{R})$ on $\Lambda^2 TN$ then gives the $\mathrm{SL}(8, \mathbb{R})$ transformations of v, σ which combine into the $\mathbf{28}$ representation.

The remaining generators of E_7 combine into a 4-form on N, $\Sigma \in \Lambda^4 T^* N$. The infinitesimal action $\mathrm{U}(\Sigma)$ of Σ on $\phi + \chi \in \Lambda^2 T^* N \oplus \Lambda^2 TN$ is

$$U(\Sigma) : \phi + \chi \mapsto \phi + \chi + \iota_\chi \Sigma + \iota_\phi * \Sigma \tag{4.12}$$

where $*\Sigma$ is the dual on N, $*\Sigma = \iota_\Sigma (k \wedge \widetilde{\epsilon}) \in \Lambda^4 TN$. The 4-form Σ gives a 3-form Θ and 4-form β' on $\mathcal{M}$, and the 4-form β' dualises to a 3-vector $\beta \in \Lambda^3 T\mathcal{M}$ (given by $\beta = *\beta' = \iota_{\beta'}\widetilde{\epsilon}$). Then the transformation (4.12) gives the infinitesimal transformation of $U = v + \rho + \sigma + \tau$ under the action of the 3-form Θ and 3-vector β. The corresponding transformations under E_n in dimensions $n < 7$ follow by truncation.

The vielbein $\mathcal{V}$ is constructed following [21], and can be parameterised in terms of G_{ij}, the 3-form C_{ijk} and a vector $B^j = \frac{1}{6!}\epsilon^{j i_1 \ldots i_6} \widetilde{C}_{i_1 \ldots i_6}$ in the factorised form

$$\mathcal{V} = \alpha \beta \exp[U(C \wedge k)] \tag{4.13}$$

Here $U(C \wedge k)$ is the map (4.12) with

$$\Sigma = C \wedge k$$

This Σ has components Σ_{IJKL} where $I = 1, \ldots, 8$ label coordinates on N which satisfy

$$(*\Sigma)^{IJKL} \Sigma_{KLMN} (*\Sigma)^{MNPQ} = 0 \tag{4.14}$$

and as a result $U(C \wedge k)$ is nilpotent,

$$[U(C \wedge k)]^4 = 0$$

Then the exponential becomes the polynomial

$$\exp[U(C \wedge k)] = 1 + U + \frac{1}{2}U^2 + \frac{1}{6}U^3 \tag{4.15}$$

and so $\mathcal{V}$ is cubic in the 3-form C. The α, β are $\mathrm{SL}(8, \mathbb{R})$ transformations acting in the $28 + 28'$ representation. Their action in the fundamental 8-dimensional representation are given by 8×8 matrices $\alpha^I{}_J, \beta^I{}_J$ which take the $(7+1) \times (7+1)$ block form

$$\alpha = \begin{pmatrix} e^a{}_j & 0 \\ 0 & e^{-1} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \delta^i{}_j & B^i \\ 0 & 1 \end{pmatrix}, \tag{4.16}$$

where $e^a{}_i$ is a vielbein for M with $e^a{}_i e^b{}_j \delta_{ab} = G_{ij}$ and $e = \det(e_i{}^a) = \sqrt{\det(G_{ij})}$. The generalised metric is then given by

$$\mathcal{H} = \mathcal{V}^t \mathcal{V}$$

and is polynomial in both C and $\widetilde{C} = *B$.

n	E_n	$\mathbf{R}$	$\mathrm{SL}(n,\mathbb{R})$ reps	$\mathcal{E}$
2	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}$	2+1	2+1	$\mathcal{E}\sim T\oplus\Lambda^2T^*$
3	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$(3,2)$	$3+3$	$\mathcal{E}\sim T\oplus\Lambda^2T^*$
4	$\mathrm{SL}(5,\mathbb{R})$	10	$4+6'$	$\mathcal{E}\sim T\oplus\Lambda^2T^*$
5	$Spin(5,5)$	16	$5+10'+1$	$\mathcal{E}\sim T\oplus\Lambda^2T^*\oplus\Lambda^5T^*$
6	$E_{6(6)}$	27(+1)	$6+15'+6(+1)$	$\mathcal{E}\sim T\oplus\Lambda^2T^*\oplus\Lambda^5T^*(\oplus\Lambda^6T)$
7	$E_{7(7)}$	56	$7+21'+21+7'$	$\mathcal{E}\sim T\oplus\Lambda^2T^*\oplus\Lambda^5T^*\oplus\Lambda^6T$

Table 2: The bundle $\mathcal{E}$ over an n-dimensional manifold $\mathcal{M}$ has fibre in the representation $\mathbf{R}$ of E_{d+1}. The decomposition into $\mathrm{SL}(n,\mathbb{R})$ representations gives a corresponding decomposition of $\mathcal{E}$.

n	E_n	$\dim(E_n)$	$\dim(E_n/H_n)$	Coset moduli
2	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}$	4	3	G
3	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	11	7 =6+1	G, C_3
4	$\mathrm{SL}(5,\mathbb{R})$	24	14 =10+4	G, C_3
5	$Spin(5,5)$	45	25 =15+10	G, C_3
6	$E_{6(6)}$	78	42=21+20+1	G, C_3, C_6
7	$E_{7(7)}$	133	70=28+35+7	G, C_3, C_6

Table 3: The U-duality groups E_n, the dimensions of the cosets E_n/H_n and the parameterisation of the cosets in terms of a metric G, a 3-form C_3 and a 6-form C_6.

5. Type M extended tangent bundles and extended spin bundles

The bundles identified in the last section are summarised in table 2, with a natural action of E_n on the fibres in the representation $\mathbf{R}$. The coset E_n/H_n is parameterised by the fields given in table 3.

As discussed in section 2, $T\oplus T^*$ strictly speaking has structure group $GL(n,\mathbb{R})$, and this can be extended by twisting with a gerbe to a generalised tangent bundle with structure group $GL(n,\mathbb{R})\ltimes\Omega^{2,cl}$, where $\Omega^{2,cl}$ is the bundle of closed 2-forms, and this preserves the Courant bracket. In section 3.1, this was generalised further to type I extended tangent bundles with structure group $O(n,n)$. This will not preserve the Courant bracket in general, but such structures are relevant for non-geometric backgrounds in string theory.

In the same way, the bundle

$$T\oplus\Lambda^2T^*\oplus\Lambda^5T^*\oplus\Lambda^6T$$

has structure group $GL(n,\mathbb{R})$. This can again be twisted with a gerbe, in a similar way to section 2.2.

Consider first $T\oplus\Lambda^2T^*$. The 3-form C can be taken to be a connection with transition functions

$$(\delta C)_{\alpha\beta}\equiv C_\beta-C_\alpha=d\lambda_{\alpha\beta}$$

for some 2-forms $\lambda_{\alpha\beta}$ on the overlaps $U_\alpha\cap U_\beta$ with consistency conditions

$$(\delta\lambda)_{\alpha\beta\delta}\equiv\lambda_{\alpha\beta}+\lambda_{\beta\gamma}+\lambda_{\gamma\alpha}=d\kappa_{\alpha\beta\gamma}$$

for 1-forms $\kappa_{\alpha\beta\gamma}$ on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$. These satisfy

$$(\delta\kappa)_{\alpha\beta\gamma\delta} \equiv \kappa_{\alpha\beta\gamma} + \kappa_{\beta\gamma\delta} + \kappa_{\gamma\delta\alpha} + \kappa_{\delta\alpha\beta} = g^{-1}_{\alpha\beta\gamma\delta} d g_{\alpha\beta\gamma\delta} \tag{5.1}$$

for some maps $g_{\alpha\beta\gamma\delta} : U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \to \mathrm{U}(1)$ from quadruple overlaps to $\mathrm{U}(1)$, which in turn satisfy

$$g_{\alpha\beta\gamma\delta} g_{\beta\gamma\delta\epsilon} g_{\gamma\delta\epsilon\alpha} g_{\delta\epsilon\alpha\beta} g_{\epsilon\alpha\beta\gamma} = \mathbb{1} \tag{5.2}$$

on quintuple overlaps. (This could be generalised to allow

$$\kappa_{\alpha\beta\gamma} + \kappa_{\beta\gamma\delta} + \kappa_{\gamma\delta\alpha} + \kappa_{\delta\alpha\beta} = d\phi_{\alpha\beta\gamma\delta} \tag{5.3}$$

for some 0-forms $\phi_{\alpha\beta\gamma\delta}$ on quadruple overlaps satisfying a consistency condition $(\delta\phi)_{\alpha\beta\gamma\delta\epsilon\eta} = c_{\alpha\beta\gamma\delta\epsilon\eta}$ on quintuple overlaps for constants $c_{\alpha\beta\gamma\delta\epsilon\eta}$ satisfying $(\delta c)_{\alpha\beta\gamma\delta\epsilon\eta\kappa} = 0$.

The $\lambda_{\alpha\beta}$ can be used to define a bundle E over M by identifying $T \oplus \Lambda^2 T^*$ on U_α with $T \oplus \Lambda^2 T^*$ on U_β by the C-field action $v + \rho \mapsto v + \rho + i_v d\lambda_{\alpha\beta}$. The fibre over a point x in M is again $T_x \oplus \Lambda^2 T^*_x$, but the transition functions are now in $GL(d,\mathbb{R}) \ltimes \Omega^{3,cl}$, where $\Omega^{3,cl}$ is the bundle of closed 3-forms. This preserves the Courant bracket on $T \oplus \Lambda^2 T^*$.

This extends to $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$, with the 6-form $\widetilde{C}$ a connection with transition functions

$$(\delta\widetilde{C})_{\alpha\beta} \equiv \widetilde{C}_\beta - \widetilde{C}_\alpha = d\widetilde{\lambda}_{\alpha\beta}$$

for some 5-forms $\widetilde{\lambda}_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$ satisfying similar consistency conditions to the above. The group E_n has a subgroup containing $GL(n,\mathbb{R})$ and transformations generated by a 3-form and (for $n = 6,7$) a 6-form, and the fibres

$$U = v + \rho + \sigma + \tau$$

can be patched together on overlaps using such transformations with closed 3-form and 6-form generators. The structure group is then generated by $GL(n,\mathbb{R})$, $\Omega^{3,cl}$ and $\Omega^{6,cl}$, where $\Omega^{p,cl}$ is the bundle of closed p-forms. The action of the 3-forms and 6-forms generates a non-trivial algebra; if $\delta_3(\Lambda)$ is the transformation generated by a closed 3-form Λ and $\delta_6(\Sigma)$ is the transformation generated by a closed 6-form Σ, then these satisfy an algebra $\mathcal{A}$ whose only non-trivial commutation relation is [26]

$$[\delta_3(\Lambda), \delta_3(\Lambda')] = \delta_6(\Lambda \wedge \Lambda')] \tag{5.4}$$

Then the structure group is $GL(n,\mathbb{R}) \ltimes \mathcal{A}$.

To incorporate non-geometric backgrounds, the bundle $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$ with transition functions in $GL(n,\mathbb{R})$ (or its generalisation twisted by gerbes with structure group $GL(n,\mathbb{R}) \ltimes (\Omega^{3,cl} \oplus \Omega^{6,cl})$) is generalised to a vector bundle $\mathcal{E}$ over the n-dimensional oriented manifold $\mathcal{M}$ with structure group E_n and fibres in the representation $\mathbf{R}$ given in table 2 for each value of n. This will be referred to as an extended tangent bundle. In general, the transition functions will mix the metric G with the gauge fields C_3, C_6, so that these will be defined locally in patches through the choice of $\mathcal{V}_\alpha$, but will not patch together to form tensor fields or gerbe connections.

n	E_n	H_n	$\widetilde{H}_n$
2	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}$	$\mathrm{SO}(2)$	$Spin(2)$
3	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	$Spin(3)\times Spin(2)$
4	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$Spin(5)$
5	$Spin(5,5)$	$(\mathrm{Sp}(2)\times\mathrm{Sp}(2))/\mathbb{Z}_2$	$\mathrm{Sp}(2)\times\mathrm{Sp}(2)$
6	$E_{6(6)}$	$\mathrm{Sp}(4)/\mathbb{Z}_2$	$\mathrm{Sp}(4)$
7	$E_{7(7)}$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{SU}(8)$
8	$E_{8(8)}$	$Spin(16)/\mathbb{Z}_2$	$Spin(16)$

Table 4: The U-duality groups E_n, their maximal compact subgroups H_n, and the double covers $\widetilde{H}_n$ of H_n.

As discussed in section 3.2, the extended tangent bundle $\mathcal{E}$ with structure group E_n can be reduced to a bundle $\bar{\mathcal{E}}$ with structure group H_n, the maximal compact subgroup of E_n given in table 1, and the reduction is equivalent to a choice of vielbein $\mathcal{V}$. The groups H_n each have a natural double cover $\widetilde{H}_n$ given in table 4. The various $\mathbb{Z}_2$ factors and double cover maps are given in [23]. An M-type extended spin bundle $\widetilde{\mathcal{E}}$ is a bundle over $\mathcal{M}$ that projects onto $\bar{\mathcal{E}}$ under the projection $p:\widetilde{H}_n\to H_n$, and a necessary and sufficient condition for this is that $w_2(\bar{\mathcal{E}})=0$.

6. Type II geometries

In this section, the generalisations of generalised geometry suggested by type II string theory on a d-dimensional manifold M are studied, in which $T\oplus T^*$ with a natural action of $\mathrm{SO}(d,d)$ is replaced by $\mathcal{E}^\pm\sim T\oplus T^*\oplus S^\pm\oplus\dots$ with a natural action of E_{d+1}. The positive chirality spin bundle S^+ is used for type IIA string backgrounds and the negative chirality spin bundle S^- is used for type IIB string backgrounds, so $\mathcal{E}^+$ will be referred to as a type IIA geometry and $\mathcal{E}^-$ will be referred to as a type IIB geometry. For a given embedding $\mathrm{SO}(d,d)\subset E_{d+1}$, the two choices of chirality give two distinct representations $\mathbf{R}^\pm$ of E_{d+1}. Equivalently, one could fix the representation $\mathbf{R}$ of E_{d+1} and choose two different embeddings $\mathrm{SO}(d,d)\subset E_{d+1}$ to obtain two decompositions $\mathcal{E}\sim T\oplus T^*\oplus S^\pm\oplus\dots$.

6.1 $d=3$, $E_4=\mathrm{SL}(5,\mathbb{R})$

Consider first the case $d=3$, with $E_4=\mathrm{SL}(5,\mathbb{R})$. For $\mathcal{E}^+$, we take the fibres to be in the $\mathbf{10}$ representation. Under the $\mathrm{SL}(4,\mathbb{R})=Spin(3,3)$ subgroup, the $\mathbf{10}$ of $\mathrm{SL}(5,\mathbb{R})$ decomposes as $\mathbf{10}\to\mathbf{6}+\mathbf{4}$, corresponding to the vector and negative chirality spinor representations of $Spin(3,3)$. Under $\mathrm{SL}(3,\mathbb{R})\subset Spin(3,3)$, the $\mathbf{6}$ decomposes into the $\mathbf{3}+\mathbf{3}'$, and the $\mathbf{4}$ decomposes into the $\mathbf{1}+\mathbf{3}$. Then locally the fibres of $\mathcal{E}^+$ decompose into $T\oplus T^*\oplus\Lambda^0T^*\oplus\Lambda^2T^*$. A section is then a formal sum

$$U=v+\xi+\rho_0+\rho_2$$

d	E_{d+1}	$\mathbf{R}$	$Spin(d,d)$ reps	$\mathcal{E}$
2	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$(3,2)$	$4+2^{\pm}$	$\mathcal{E}\sim T\oplus T^*\oplus S^{\pm}$
3	$\mathrm{SL}(5,\mathbb{R})$	10	$6+4^{\pm}$	$\mathcal{E}\sim T\oplus T^*\oplus S^{\pm}$
4	$Spin(5,5)$	16	$8+8^{\pm}$	$\mathcal{E}\sim T\oplus T^*\oplus S^{\pm}$
5	$E_{6(6)}$	$27(+1)$	$10+1(+1)+4^{\pm}$	$\mathcal{E}\sim T\oplus T^*\oplus\Lambda^5T^*(\oplus\Lambda^5T)\oplus S^{\pm}$
6	$E_{7(7)}$	56	$12+12+32^{\pm}$	$\mathcal{E}\sim T\oplus T^*\oplus\Lambda^5T\oplus\Lambda^5T^*\oplus S^{\pm}$

Table 5: The bundle $\mathcal{E}$ over a d-dimensional manifold M has fibre in the representation $\mathbf{R}$ of E_{d+1}. The decomposition into $Spin(d,d)$ representations gives a corresponding decomposition of $\mathcal{E}$. The upper sign is for the IIA geometry and the lower one for IIB geometry.

of a vector v, a 1-form ξ, a 0-form ρ_0 and a 2-form ρ_2. The 0-form ρ_0 and 2-form ρ_2 combine to form a positive chirality spinor of $Spin(3,3)$, so that U is the sum of a vector $V=v+\xi$ and a spinor $\rho^+=\rho_0+\rho_2$ of $Spin(3,3)$.

Similarly, for $\mathcal{E}^-$, we take the fibres to be in the dual $\mathbf{10}'$ representation, decomposing as $\mathbf{10}'\to\mathbf{6}+\mathbf{4}'$ under $\mathrm{SL}(4,\mathbb{R})=Spin(3,3)$, and further decomposing into $\mathrm{SL}(3,\mathbb{R})$ representations gives $\mathbf{3}+\mathbf{3}'+\mathbf{1}+\mathbf{3}'$. Then a section is a formal sum

$$U=v+\xi+\rho_1+\rho_3$$

of a vector v, a 1-form ξ, a 1-form ρ_1 and a 3-form ρ_3, in

$$T\oplus T^*\oplus T^*\oplus\Lambda^3T^*\sim T\oplus T^*\oplus S^-$$

The 1-form ρ_1 and 3-form ρ_3 combine to form a negative chirality spinor of $Spin(3,3)$, so that U is the sum of a vector $V=v+\xi$ and a spinor $\rho^-=\rho_1+\rho_3$ of $Spin(3,3)$.

The adjoint of $\mathrm{SL}(5,\mathbb{R})$ decomposes under $Spin(3,3)$ as

$$24=15+1+4^++4^- \tag{6.1}$$

with two spinor generators $\Theta^{\pm}\in S^{\pm}$. In addition to the standard action of $Spin(3,3)$ and a scaling transformation, there are two extra generators in spin representations of $Spin(3,3)$ that transform $T\oplus T^*$ and $S^{\pm}$ into one another. The coset space $\mathrm{SL}(5,\mathbb{R})/\,\mathrm{SO}(5)$ is 14-dimensional and can be parameterised by a metric G_{ij}, 2-form B_{ij} and scalar Φ, together with either even forms C_0,C_2 combining into a positive chirality spinor C^+, or odd forms C_1,C_3 combining into a positive chirality spinor C^-. These two possibilities correspond to two gauge choices for the local SO(5). The parametrisation in terms of C^+ is useful for the IIA string and that in terms of C^- for the IIB string. The generators $\Theta^{\pm}$ act as shifts of $C^{\pm}$,

$$C^{\pm}\mapsto C^{\pm}+\Theta^{\pm}$$

6.2 General $d\le 6$

A similar structure applies for other $d\le 6$, as summarised in table 3. For $d=2,3,4$, $T\oplus T^*$ is extended to

$$\mathcal{E}^{\pm}=T\oplus T^*\oplus S^{\pm}$$

with a natural action of E_{d+1}. For example, for $d = 4$, the fibre is in the positive chirality spinor representation 16^+ of $E_5 = Spin(5,5)$ for the IIA geometry. Under the natural embedding of $Spin(4,4) \subset Spin(5,5)$, the 16^+ decomposes into the spinor representations $8^+ + 8^-$ of $Spin(4,4)$. This is related by $Spin(4,4)$ triality to an embedding in which it decomposes into a vector and spinor $8_v + 8^+$, and this is the embedding used here, with $\mathcal{E}^+ \sim T \oplus T^* \oplus S^+$. For type IIB, $\mathcal{E} \sim T \oplus T^* \oplus S^-$, and this can either be regarded as coming from the same embedding of $Spin(4,4) \subset Spin(5,5)$ but with the fibres in the negative chirality spinor representation 16^- of $Spin(5,5)$, or as arising from keeping the same 16^+ representation but choosing a different embedding of $Spin(4,4) \subset Spin(5,5)$ (related by triality to the other two embeddings discussed above).

For $d = 5, 6$, $T \oplus T^*$ is extended to

$$\mathcal{E} \sim T \oplus T^* \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus S^{\pm}$$

transforming under E_d, with $\Lambda^5 T$ corresponding to NS 5-brane charge and $\Lambda^5 T^*$ corresponding to KK monopole charge. For $d = 5$, this is the reducible $27 + 1$ representation, and the $\Lambda^5 T$ factor can be removed to leave the 27. For $d = 6$, E_7 has a maximal subgroup $\mathrm{SO}(6,6) \times \mathrm{SL}(2,\mathbb{R})$, and under this the 56 decomposes as $56 = (12,2) + (32,1)$. As $\Lambda^5 T \oplus \Lambda^5 T^* \sim T^* \oplus T$ for $d = 6$,

$$\mathcal{E} \sim T \oplus T^* \oplus T \oplus T^* \oplus S^{\pm}$$

and $T \oplus T^*$ forms an $\mathrm{SL}(2,\mathbb{R})$ doublet with $\Lambda^5 T \oplus \Lambda^5 T^*$, with both in the 12-dimensional vector representation of $\mathrm{SO}(6,6)$.

The decomposition of the adjoint of E_{d+1} into $Spin(d,d)$ representations is given in table 5. In each case there are two spinor generators, which are of the same chirality for d even and opposite chiralities for odd d. Convenient parameterisations of the coset space E_{d+1}/H_{d+1} are also given. For each d, these are represented by

$$G, B, \widetilde{B}, \Phi, C^{\mp}$$

including the d^2 parameters assembled into the metric G and 2-form B, a scalar Φ, and a 6-form $\widetilde{B}$ which only contributes for $d = 6$, corresponding to a 6-form field dual to the 2-form B. In addition, for the IIA theory there is a negative chirality spinor C^- corresponding to a set of odd forms $C^- \sim C_1, C_3, C_5$, while for the IIB theory there is a positive chirality spinor C^- corresponding to a set of even forms $C^+ \sim C_0, C_2, C_4, C_6$.

6.3 Reduction of M-geometries to type IIA geometries

Consider an M-geometry on an n-dimensional manifold $\mathcal{M}$ which is a circle bundle over a $d = n-1$-dimensional manifold M, with $n \leq 7$. As in subsection 4.4, each p-form on $\mathcal{M}$ that is invariant under the circle action projects to a p-form and a $p-1$-form on M, and each invariant p-vector on $\mathcal{H}$ projects to a p-vector and a $p-1$-vector on M. Thus

$$\Lambda^p T\mathcal{M}|_{\mathrm{U}(1)} \sim \Lambda^p TM \oplus \Lambda^{p-1} TM, \qquad \Lambda^p T^*\mathcal{M}|_{\mathrm{U}(1)} \sim \Lambda^p T^* M \oplus \Lambda^{p-1} T^* M \tag{6.2}$$

d	E_{d+1}	Adjoint	$Spin(d,d)$ decomposition	E_{d+1}/H_{d+1} parameterisation
2	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$8+3$	$6+1+2^{\mp}+2^{\mp}$	$7 = 6+1+2^{\mp}$
3	$\mathrm{SL}(5,\mathbb{R})$	24	$15+1+4^{+}+4^{-}$	$14 = 9+1+4^{\mp}$
4	$Spin(5,5)$	45	$28+1+8^{\mp}+8^{\mp}$	$25 = 16+1+8^{\mp}$
5	$E_{6(6)}$	78	$45+1+16^{+}+16^{-}$	$42 = 25+1+16^{\mp}$
6	$E_{7(7)}$	133	$66+1+1+1+32^{\mp}+32^{\mp}$	$70 = 36+1+1+32^{\mp}$

Table 6: The bundle $\mathcal{E}$ over a d-dimensional manifold M has fibre in the representation $\mathbf{R}$ of E_{d+1}. The decomposition into $Spin(d,d)$ representations gives a corresponding decomposition of $\mathcal{E}$.

The M-geometry on $\mathcal{M}$ is based on

$$T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$$

For invariant forms and multi-vectors, this reduces to the following structure on M:

$$T \oplus T^* \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus [\Lambda^0 T^* \oplus \Lambda^2 T^* \oplus \Lambda^4 T^* \oplus \Lambda^6 T^*] \sim T \oplus T^* \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus S^+ \quad (6.3)$$

This uses that for $d = 6$, $\Lambda^6 TM \sim \Lambda^6 T^* M$, while for $d < 6$, $\Lambda^6 TM$ does not arise.

The generalised metric on $\mathcal{M}$ is parameterised by a metric G, a 3-form C and a 6-form $\widetilde{C}$. If these are invariant under the circle action, then the 3-form projects to a 2-form B and 3-form C_3, the 6-form $\widetilde{C}$ gives a 6-form $\widetilde{B}$ and a 5-form C_5, while the metric projects to a metric G_M, 1-form C_1 and scalar Φ on M. In this way the M-geometry generalised metric $\mathcal{H}(G, C, \widetilde{C})$ on $\mathcal{M}$ gives rise to the IIA-geometry generalised metric $\mathcal{H}(G_M, B, \widetilde{B}, \Phi, C_1, C_3, C_5)$ depending on the IIA-geometry fields on a manifold of dimension $d \leq 6$:

$$\{G_M, B, \widetilde{B}, \Phi, C_1, C_3, C_5\} \sim \{G_M, B, \widetilde{B}, \Phi, S^-\} \quad (6.4)$$

The explicit parameterisation of the M-geometry generalised metric $\mathcal{H}$ on a 7-fold $\mathcal{M}$ given in subsection 4.4 then gives that of the type-IIA generalised metric on a 6-fold M, and the parameterisation of the type-IIA generalised metric for $d < 6$ follows by truncation.

7. Type II extended tangent bundles and extended spin bundles

The type II geometries have extended tangent spaces of the form $\mathcal{E} \sim T \oplus T^* \oplus \Lambda^{\pm}$ or

$$\mathcal{E} \sim T \oplus T^* \oplus \Lambda^5 T \oplus \Lambda^5 T^* \oplus \Lambda^{\pm}$$

where $\Lambda^{\pm}$ are the bundles of even or odd forms, and these have structure group $GL(d,\mathbb{R})$. The coset space is parameterised by the fields $G, B, \widetilde{B}, \Phi, C^{\mp}$. This structure can be twisted by gerbes by allowing the p-form fields to be gauge fields with transition functions that are closed p-forms. The action of E_{d+1} includes transformations generated by p-forms for the same values of p that act as shifts of the p-form gauge fields and so can be used in the transition functions in the same way as in section 5. As in section 5, this can be generalised to allow general vector bundles with structure group E_{d+1} and with fibres in the representations $\mathbf{R}^{\pm}$ given in table 5. Again, this will give a non-geometric construction in general, as the transition functions will mix the metric with the various gauge fields. The type II extended tangent bundle $\mathcal{E}_{d+1}$ over a d-manifold M with structure group E_{d+1} reduces to a bundle $\bar{\mathcal{E}}$ with compact structure group H_{d+1}, and a type II generalised spin-bundle is a bundle $\widetilde{\mathcal{E}}$ with structure group $\widetilde{H}_{d+1}$, the double cover of H_{d+1} from table 4, that projects onto $\bar{\mathcal{E}}$ under the double cover map.

8. Special structures, generalised holonomy and supersymmetry

In Riemannian geometry, interesting classes of geometry are characterised by specifying the holonomy of the Levi-Civita connection. In n dimensions, a general space will have holonomy $O(n)$, but a Kahler space has holonomy $U(n/2)$ (for n even), a Calabi-Yau space has holonomy $\mathrm{SU}(n/2)$ (for n even), and special holonomies G_2 and $Spin(7)$ can arise for $n = 7, 8$ respectively. There is an intimate relation between the holonomy and the number of covariantly constant spinors, and hence the number of supersymmetries preserved when the geometry is used in a supergravity solution.

In generalised geometry, interesting classes are given by generalised complex [1], generalised Kahler [4] and generalised Calabi-Yau geometries [1], and these too are related to supersymmetry [5]–[20].

In previous sections, extended tangent and spin bundles of types I,II and M were discussed, and geometries specified by a metric G and various antisymmetric tensor fields. In this section, connections on the extended spin bundle that are constructed from this geometrical data will be discussed, and interesting restrictions on the geometry defined by restricting the holonomies of these connections.

8.1 Generalised holonomy in generalised geometry and type I extended geometry

As was seen in section 3.1, a type I extended tangent bundle is a bundle E over a d-dimensional space M with structure group $O(d,d)$ (or $\mathrm{SO}(d,d)$), and reduces to a bundle $E^+ \oplus E^-$ with structure group $O(d) \times O(d)$ or $S(O(d) \times O(d))$, and each sub-bundle is isomorphic to the tangent bundle, $E^{\pm} \sim T$ [4].

Consider first the case in which the extended geometry is a generalised geometry, which will be the case if the structure group of E is in $GL(d, \mathbb{R}) \ltimes \Omega^{2,cl}$, so that for $E^+ \oplus E^-$ it is in the diagonal $O(d) \subset O(d) \times O(d)$. A generalised metric corresponds to a metric G and closed 3-form H on M, with $H = dB$ for some 2-form gerbe connection B. Let $\nabla^{\pm}$ be the metric connection on T given by the Levi-Civita connection plus torsion $\pm\frac{1}{2}G^{-1}H$, $\nabla^{\pm} = \nabla \pm \frac{1}{2}G^{-1}H$, so that

$$\nabla_i^{\pm} v^j = \nabla_i v^j \pm \frac{1}{2} H^j_{ik} v^k \tag{8.1}$$

where $H^j_{ik} = H_{ikl}G^{lj}$.

The holonomies of these connections, $\mathcal{H}^{\pm} = \mathcal{H}(\nabla^{\pm})$, are in $O(d)$, $\mathcal{H}^{\pm} \subseteq O(d)$. If $d = 2m$ and $\mathcal{H}^+ \subseteq U(m)$, then there is an almost complex structure J^+ that is parallel with respect to ∇^+, $\nabla^+ J^+ = 0$. Similarly, if $\mathcal{H}^- \subseteq U(m)$ there is an almost complex

(p,q)	$\mathcal{H}^+$	$\mathcal{H}^-$	dimension
$(1,1)$	$O(d)$	$O(d)$	d
$(2,1)$	$U(m)$	$O(2m)$	$2m$
$(2,2)$	$U(m)$	$U(m)$	$2m$
$(4,1)$	$\mathrm{Sp}(M)$	$O(4M)$	$4M$
$(4,2)$	$\mathrm{Sp}(M)$	$U(2M)$	$4M$
$(4,4)$	$\mathrm{Sp}(M)$	$\mathrm{Sp}(M)$	$4M$

Table 7: The holonomies $\mathcal{H}^+$, $\mathcal{H}^-$ giving $p-1$ complex structures J^+ and $q-1$ complex structures J^- for manifolds of various dimension, which allow the construction of sigma-models with (p,q) supersymmetry.

structure J^- with $\nabla^- J^- = 0$. The metric is hermitian with respect to each structure. An interesting case is that in which $\mathcal{H}^\pm \subseteq U(m)$, and this gives precisely the geometry needed to define a sigma-model with (2,2) world-sheet supersymmetry [45]. The superalgebra closes off-shell if both $J^\pm$ are integrable, and this gives precisely the bihermitian geometry of [45] which has been termed generalised Kahler geometry in [4].

The isomorphism $E^\pm \sim T$ then gives corresponding connections $\nabla^\pm$ on $E^\pm$, and the connection with supersymmetry suggests using the connection ∇^+ on E^+ and the connection ∇^- on E^-. Then the almost complex structures $J^\pm$ on T correspond to generalised almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ on E, and if $J^\pm$ are integrable, then $\mathcal{J}_1, \mathcal{J}_2$ are Courant-integrable and so are generalised almost complex structures [4].

There is a similar story for other holonomy groups [46, 47]. In table 7, the holonomy groups $\mathcal{H}^\pm$ that give sigma-models with (p,q) supersymmetry are given. (The cases (q,p) are given by interchanging $\mathcal{H}^+, \mathcal{H}^-$.)

In each case, there are $p-1$ almost complex structures J^+_α, $\alpha = 1, \ldots, p-1$ satisfying $\nabla^+ J^+_\alpha = 0$, and $q-1$ almost complex structures $J^+_{\alpha'}$, $\alpha' = 1, \ldots, q-1$ satisfying $\nabla^- J^-_{\alpha'} = 0$. If there are three J^+ or J^-, they satisfy the quaternion algebra and so constitute an almost quaternionic structure. Each pair $(J^+_\alpha, J^-_{\alpha'})$ defines two generalised almost complex structures $\mathcal{J}_1^{\alpha\alpha'}, \mathcal{J}_2^{\alpha\alpha'}$ as in [4], giving $2(p-1)(q-1)$ generalised almost complex structures. For the $(4,2)$ case, there are 3+3 generalised almost complex structures $\mathcal{J}_1^\alpha, \mathcal{J}_2^\alpha$ satisfying an algebra with e.g.

$$[\mathcal{J}_1^\alpha, \mathcal{J}_1^\beta] = [\mathcal{J}_2^\alpha, \mathcal{J}_2^\beta] = \epsilon^{\alpha\beta\gamma} \mathcal{J}_1^\gamma \Pi^+ \tag{8.2}$$

where $\Pi^\pm$ is the projection $\Pi^\pm : E \to E^\pm$. For the $(4,4)$ case, there are 9+9 generalised almost complex structures $\mathcal{J}_1^{\alpha\alpha'}, \mathcal{J}_2^{\alpha\alpha'}$. If all the almost complex structures are integrable, then the space is generalised Kahler if $p \geq 2$ and $q \geq 2$. It seems natural to refer to the (4,4) case [45] as generalised hyperkahler, as in [54, 55].

The connections $\nabla^\pm$ on T lift to connections on the spin bundle (assuming M is spin), with

$$\tilde{\nabla}^\pm_i \alpha = \nabla_i \alpha \pm \frac{1}{8} H_{ijk} \Gamma^{jk} \alpha \tag{8.3}$$

for spinors α, where $\Gamma^{ij} = \Gamma^{[i}\Gamma^{j]}$ and Γ^i satisfy the Clifford algebra

$$\{\Gamma^i, \Gamma^j\} = 2G^{ij}\mathbb{1} \tag{8.4}$$

The holonomies $\widetilde{\mathcal{H}}^{\pm}$ of $\widetilde{\nabla}^{\pm}$ are in $Spin(d)$ and determine the number of covariantly constant spinors $\alpha^{\pm}$ satisfying $\widetilde{\nabla}^{\pm}\alpha^{\pm} = 0$ [46]. For general holonomy $\widetilde{\mathcal{H}}^{+} = Spin(d)$, there are no covariantly constant spinors, while if $d = 2m$ and $\widetilde{\mathcal{H}}^{+} \subseteq \mathrm{SU}(m)$, then there are at least two satisfying $\widetilde{\nabla}^{+}\alpha^{+} = 0$. The relation between holonomy and the number of parallel spinors is well-known: for example, for $d = 8$, there will be $1, 2, 3$ or 4 such spinors for holonomies $Spin(7), \mathrm{SU}(4), \mathrm{Sp}(2), \mathrm{SU}(2) \times \mathrm{SU}(2)$ respectively, while for $d = 7$, there is one such spinor for holonomy G_2.

Similar results apply for type I extended geometries. A bundle E with $O(d, d)$ structure reduces to a bundle $E^{+} \oplus E^{-}$ with structure group $O(d) \times O(d)$. In special cases, this will be reducible, and in this extended case, the structure group of E^{+} need not be the same as that for E^{-}. The connections with torsion $\nabla^{\pm}$ again give connections on $E^{\pm}$, and we choose the connection ∇^{+} on E^{+} and ∇^{-} on E^{-}. Again, there are interesting geometries with restrictions on the holonomies $\mathcal{H}^{\pm}$. For $d = 2m$, bundles with $\mathcal{H}^{+} \times \mathcal{H}^{-}$ in $U(m) \times U(m)$ will be referred to as extended Kahler, and bundles with $\mathcal{H}^{+} \times \mathcal{H}^{-}$ in $\mathrm{SU}(m) \times \mathrm{SU}(m)$ will be referred to as extended Calabi-Yau. The connections again lift to connections on the extended spin bundle with structure $Spin(d) \times Spin(d)$, and the number of covariantly constant sections of these bundles play an important role in understanding supersymmetry in non-geometric backgrounds, as will be discussed elsewhere.

8.2 Generalised holonomy in generalised geometry and M-extended geometry

For an M-geometry on an n-dimensional manifold $\mathcal{H}$, the extended tangent bundle $\mathcal{E}$ has an E_n-structure and is reducible to one with compact structure group H_n, while the extended spin bundle $\widetilde{\mathcal{E}}$ has structure $\widetilde{H}_n$. For a conventional geometry, the structure groups reduce further to $\mathrm{SO}(n)$ and $Spin(n)$ respectively, while the more general cases are relevant to non-geometric backgrounds.

Consider first the case of conventional geometry. Sections of $\widetilde{\mathcal{E}}$ are then spinor fields on $\mathcal{H}$, and there is a natural connection on $\widetilde{\mathcal{E}}$ that generalises (8.1), given by

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{24}\Gamma^{jkl}F_{ijkl} \tag{8.5}$$

where $F = dC$, ∇_i is the usual spin connection, Γ_i are Dirac matrices and $\Gamma_{ij\ldots k}$ are antisymmetrised products of gamma matrices. Note that, unlike (8.1), this does not project onto a connection on the tangent bundle. Remarkably, this connection has holonomy $\mathcal{H}$ that is always contained in $\widetilde{H}_n$ [42]. Interesting geometries arise when the holonomy is a special subgroup of $\widetilde{H}_n$.

This generalises to the case when the extended spin bundle is not reducible to the spin bundle, so that the structure group is in $\widetilde{H}_n$, and sections are not spinor fields. The derivative (8.5) lifts to one acting on $\widetilde{\mathcal{E}}$, and again the holonomy is in $\widetilde{H}_n$.

8.3 Seven-dimensional spaces

Consider the case in which $\mathcal{M}$ is seven-dimensional, $n = 7$. For a Riemannian space with metric G, the holonomy $\mathcal{H}(\nabla)$ of the Levi-Civita connection is in SO(7), There will be at least one covariantly constant spinor satisfying $\nabla\alpha = 0$ provided the holonomy is in G_2, $\mathcal{H}(\nabla) \subseteq G_2$.

For the extended spin bundle, the holonomy $\mathcal{H}$ of the connection (8.5) is in $\widetilde{H}_7 = \mathrm{SU}(8)$. There will be at least one section of $\widetilde{\mathcal{E}}$ that is covariantly constant with respect to the connection (8.5) provided the holonomy is in the subgroup of SU(8) preserving an element α transforming in the **8** of SU(8), $\mathcal{H} \subseteq \mathrm{U}(7) \ltimes \mathbb{C}^7$.

8.4 Relation with supersymmetry

For type I backgrounds, Killing spinors are spinors α^+, α^- that are covariantly constant

$$\widetilde{\nabla}^{\pm}\alpha^{\pm} = 0 \tag{8.6}$$

and for which in addition there is a scalar Φ such that

$$\frac{1}{6} H_{ijk}\Gamma^{ijk}\alpha^{\pm} = \pm(\partial_i\Phi)\Gamma^i\alpha^{\pm} \tag{8.7}$$

The bosonic fields of 11-dimensional supergravity are a metric G_{MN} and a 3-form gauge field C_{MNP} $(M, N = 0, 1, \ldots, 10)$, with a vielbein $e_M{}^A$ satisfying $e_M{}^A e_N{}^B \eta_{AB} = G_{MN}$ used to convert coordinate indices M, N to tangent space indices A, B. The supercovariant derivative acting on spinors is

$$\hat{\nabla}_M = \nabla_M - \frac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})F_{NPQR}, \tag{8.8}$$

where $F = dC$, the Γ_A are $D = 11$ Dirac matrices and $\Gamma_{AB\ldots C}$ are antisymmetrised products of gamma matrices, $\Gamma_{AB\ldots C} = \Gamma_{[A}\Gamma_B \ldots \Gamma_{C]}$. The signature is $(- + + \cdots +)$, and ∇_M is the usual Riemannian covariant derivative involving the Levi-Civita connection ω_M taking values in the tangent space group $Spin(10, 1)$

$$\nabla_M = \partial_M + \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB}. \tag{8.9}$$

Each solution of

$$\hat{\nabla}_M\epsilon = 0, \tag{8.10}$$

is a Killing spinor field that generates a supersymmetry leaving the background invariant, so that the number of supersymmetries preserved by a supergravity background depends on the number of supercovariantly constant spinors satisfying (8.10). Any commuting Killing spinor field ϵ defines a Killing vector $v_A = \bar{\epsilon}\Gamma_A\epsilon$, which is either timelike or null, together with a 2-form $\bar{\epsilon}\Gamma_{AB}\epsilon$ and a 5-form $\bar{\epsilon}\Gamma_{ABCDE}\epsilon$.

The integrability conditions for (8.10) are satisfied if the background satisfies the supergravity field equations

$$R_{MN} = \frac{1}{12}\left(F_{MPQR}F_N{}^{PQR} - \frac{1}{12}g_{MN}F^{PQRS}F_{PQRS}\right) \tag{8.11}$$

and

$$d*F + \frac{1}{2}F \wedge F = 0, \tag{8.12}$$

but the integrability conditions are weaker than the field equations.

Let

$$f = \frac{1}{24}F_{MPQR}\Gamma^{MPQR} \tag{8.13}$$

and note that the derivative (8.8) can be rewritten as

$$\tilde{\nabla}_M = \nabla_M + \frac{1}{24}\Gamma^{PQR}F_{MPQR} - \frac{1}{12}\Gamma_M f \tag{8.14}$$

Then for backgrounds in which the Killing spinor satisfies

$$f\epsilon = 0 \tag{8.15}$$

(such a constraint was used in [51−53, 42]) the Killing spinor condition simplifies to

$$\tilde{\tilde{\nabla}}_M\epsilon \equiv (\nabla_M + \frac{1}{24}\Gamma^{PQR}F_{MPQR})\epsilon = 0 \tag{8.16}$$

and the analysis of supersymmetric backgrounds in terms of the holonomy $\mathcal{H}(\hat{\nabla})$ [42].

Consider product spaces $\mathcal{M} = M_{\widetilde{n}} \times M_n$ of spaces of dimensions $n, \widetilde{n} = 11 - n$, so that the coordinates can be split into x^μ, y^i with $\mu, \nu = 1, \ldots, \widetilde{n} = 11 - n$ and $i, j = 1, \ldots, n$, with a product metric of the form

$$G_{MN} = \begin{pmatrix} G_{\mu\nu}(x) & 0 \\ 0 & G_{ij}(y) \end{pmatrix} \tag{8.17}$$

where $g_{\mu\nu}(x)$ has Lorentzian signature and $g_{ij}(y)$ has Euclidean signature. A convenient realisation of the gamma matrices Γ_M in terms of the gamma matrices γ_μ on $M_{\widetilde{n}}$ and the ones $\widetilde{\Gamma}_i$ on M_n is, for n even,

$$\Gamma_\mu = \gamma_\mu \otimes \widetilde{\Gamma}_*, \qquad \Gamma_i = 1 \otimes \widetilde{\Gamma}_i \tag{8.18}$$

where $\widetilde{\Gamma}_*$ is the chirality operator on $M_{\widetilde{d}}$, $\widetilde{\Gamma}_* \propto \prod_i \widetilde{\Gamma}_i$. There is a similar realisation for n odd. A spinor ϵ decomposes as $\epsilon = \eta \otimes \alpha$ where η is a spinor on $M_{\widetilde{n}}$ and α is a spinor on M_n.

Suppose $M_{\widetilde{n}}$ is $\widetilde{n}$ dimensional Minkowski space with flat metric $G_{\mu\nu}$, and the only non-vanishing components of F are F_{ijkl} in the 'internal space' M_n. Then for any spinor α on M_n satisfying

$$\tilde{\nabla}_i\alpha = 0 \tag{8.19}$$

where

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{24}\Gamma^{jkl}F_{ijkl} \tag{8.20}$$

and the condition

$$F_{ijkl}\Gamma^{ijkl}\alpha = 0 \tag{8.21}$$

there will be a Killing spinor satisfying (8.10) of the form $\eta \otimes \alpha$ where η is any (covariantly) constant spinor in Minkowski space. Thus supersymmetric backgrounds arise when the connection $\widetilde{\nabla}$ has a special holonomy so that there are solutions of (8.19), and in addition each solution satisfies (8.21).

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